Kuratowski’s Closure-Complement problem

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October 6, 2010

A famous theorem in point-set topology due to Kuratowski was stated as an assignment problem in our topology class. With a hint from the prof, I solved the problem which I regard as the best problem in point-set topology I ever solved yet. The problem seems too fascinating to be even true. It appeared first in a paper due to Kuratowski\(^1\) and was made popular by Kelley in his book on *General Topology*\(^2\) and can be found as a “starred” problem in Munkres’ *Topology*. I could solve it, not mainly because of the hint given by the prof but with the comfort that someone has already solved the problem. Without further ado, I state the problem:

**Theorem 0.1** Let \((X, \tau)\) be a topological space and \(A \subseteq X\). By iteratively applying operations of closure and complementation, one cannot produce more than 14 disjoint sets.

*Prima facie*, there does not seem to be a reason that this process should stop at all. But call it the ‘smart’ way a topology was defined or whatever, this process stops, and gives one no more than 14 different sets. The bound 14 is the best possible, one *does* have an “easy” subset of the real line where this bound is achieved.

**Proof:** We first make an observation that may seem to be a digression but is very much related to the problem at hand.

**Lemma 0.1** Let \(S = \{x, y\}\) and \(M(S)\) be the free monoid generated by \(S\). Let \(R\) be the relations \(\{x^2 = x, y^2 = 1, yxyxyxy = yxy\}\). Then, \(M = \langle M(S)|R \rangle\) is a monoid with 14 elements.

**Proof:** Since \(x^2 = x\) and \(y^2 = 1\), we can safely assume that the words of our monoid have no powers of \(x\) and \(y\). Thus the only nontrivial words that survive this collapsing are \(x, y, xy, yx, yx, yxy, yxy, \ldots\). The relation \(yxyxyxy = yxy\) restricts the length of any word of \(M(S)\) to 7, the unique word of length 7 being \(yxyxyxy\). Thus,

\[M = \{1, x, y, xy, yx, yxy, xyx, yxyx, yxxy, yxyxy, yxxyx, yxxyxy, yxxyxyx\}\]

By induction on the length of the word, one sees that no further reduction is possible. \(\blacksquare\)

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One observes that the operation of taking closure is idempotent and complementation is an involution. (This was the hint given).

If \( x \) denotes “taking closure”, then for any \( A \in \tau \), \( x(x(A)) = \overline{A} = \overline{A} = x(A) \). Similarly, if \( y \) denotes “complementing”, then \( y(A) = A^c \). The set map \( S = \{x, y\} \to \text{Perm}(\tau) \) on the group of permutations of \( \tau \) induces a monoid homomorphism \( \mathcal{M}(S) \to \text{Perm}(\tau) \). (Of course we are interested in a very small subset of \( \text{Perm}(\tau) \), namely the orbits of an \( A \in \tau \)). The corresponding monoid action is described below:

\[
\mathcal{M}(S) \times \tau \to \tau \\
x(A) = \overline{A} \\
y(A) = A^c.
\]

Instead of taking \( \tau \), we can restrict our action to the orbit of \( A \in \tau \), namely \( O_A \). In order to prove the theorem, we ought to show that \( |O_A| \leq 14 \).

Since \( x^2 = x \) and \( y^2 = 1 \), it only remains to show that the third relation in \( R \) of Lemma 0.1 holds. It is here that topology comes into picture. The relation \( yxyxyxy = yxy \) follows from the following two lemmas:

**Lemma 0.2** For any subset \( A \) of \( X \), \( A^0 = (A^c)^c \).

**Proof:** Let \( a \in A^0 \). So there is an open set \( U \) such that \( a \in U \subseteq A^0 \). \( U \subseteq A^0 \subseteq A \Rightarrow A^c \subseteq U^c \). But \( U^c \) is closed so \( \overline{A} \subseteq U^c \). Taking complement again, \( \overline{U} \subseteq A^c \). If \( a \in (A^c)^c \), then there is a neighbourhood \( U \) of \( a \) such that \( U \subseteq (A^c)^c \) so \( \overline{A} \subseteq U^c \). But now, \( A^c \subseteq \overline{A} \) implies that \( A^c \subseteq U^c \), or equivalently, \( U \subseteq A \). Since \( A^0 \) is the largest open subset of \( A \), \( a \in U \in A^0 \). \( \blacksquare \)

**Lemma 0.3** For any open set \( U \) of \( X \), \( ((U)^c)^c = U \).

**Proof:** \( U \subseteq U \Rightarrow (U)^c \subseteq U^c \Rightarrow (U)^c \subseteq U^c \Rightarrow U \subseteq ((U)^c)^c \). Suppose the other containment is not true, there is \( x \in U \cap (U)^c \). Thus every neighbourhood of \( x \) and in particular, \( U \), intersects \( (U)^c \). This is a contradiction. \( \blacksquare \)

To end the proof of the theorem, we observe by Lemma 0.2 that any \( A \in \tau \) can be “opened” by applying \( yxy \), that is, \( yxy(A) = A^0 \). By Lemma 0.3, \( yxyx(U) = U \) for every open set \( U \). Combining both reductions gives

\[
yxyx(yxy)(A) = (yxy)(A),
\]

justifying the third relation in \( R \) of Lemma 0.1. \( \blacksquare \)

One verifies that the subset \( (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, 5] \cap \mathbb{Q}) \) of \( \mathbb{R} \) is an example that the upper bound 14 is attained.